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Hamilton–Jacobi Treatment of Chiral Schwinger Model

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We investigate the path integral quantization of the bosonic chiral Schwinger model using multi-Hamilton–Jacobi procedure. The integrability conditions require the extension of the initial phase space. The Wess–Zumino term was recovered calculating the action corresponding to the extended system.

1. INTRODUCTION

The application of Hamilton–Jacobi formalism to field theory started with *Carathéodory*'s pioneering work (Carathéodory, 1922, 1929). Constrained dynamical systems, field theory (Bergmann, 1966; Dominici *et al.*, 1984; Goldberg *et al.*, 1991; Kastrup, 1977; Kuchar, 1982), and strings and p-branes (Hosotani and Nakayama, 1999; Kastrup, 1979; Kastrup and Rinke, 1981; Nambu, 1980, 1981) were also investigated afterwards.

Recently, singular systems with higher order Lagrangians, systems which have elements of the Berezin algebra (Pimentel *et al.*, 1996, 1998; Pimentel and Teixeira, 1998), the quantization of Proca's model (Baleanu and Güler, 2000), the nonrelativistic particle on a curved space (Baleanu and Güler, 2001), as well as supersymmetric quantum mechanics (Baleanu and Güler, in press) in Witten's version (Witten, 1981), were investigated using multi-Hamilton–Jacobi formulation initiated in Güler (1987a,b, 1989, 1992a,b).

On the other hand in Güler (1998) all local classical field theories described by the Lagrangian $L(\phi_i, \frac{\partial \phi_i}{\partial x_\mu})$, $i = 1, \ldots, n$ are treated as singular systems with

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constraints $H_0' = p_0 - L = 0$, $H_\mu' = p_\mu + \pi_i \frac{\partial \phi_i}{\partial x^\mu} = 0$, where p_0, p_μ, and π_i are generalized momenta corresponding to τ, x_μ, and ϕ_i respectively. The canonical equations corresponding to above Hamiltonians are integrable if and only if all the variations of L are zero.

For a second-class constrained system in Dirac's classification (Dirac, 1964; Hanson *et al.*, 1976; Henneaux, 1985; Henneaux and Teiteilbom, 1992; Sahbanov, 2000; Sundermeyer, 1982), like the bosonic chiral Schwinger model, the corresponding Hamilton–Jacobi equations are not integrable.

To avoid this problem we have two basic possibilities, the first one is to enlarge the phase space (Batalin and Fradkin, 1986, 1987; Batalin *et al.*, 1989a,b; Batalin and Tyutin, 1991) and the other one is to keep to the original phase space itself (Rajaraman and Mitra, 1990a,b; Rabei and Güler, 1992).

On the other hand the Hamilton–Jacobi formalism for fields requires a special attention when all Hamiltonians are densities because the surface terms (Henneaux *et al.*, 1992) play an important role in the process of quantization.

The theories in which chiral fermions interact with gauge fields possess anomalies in the fermionic current. This current can be shown to be covariantly conserved using the equations of motion but quantum effects are expected to destroy this symmetry. It has been proposed by introducing a new dynamical field contained in the Wess–Zumino action in order to render the theory gauge invariant (Faddeev and Shatashvili, 1986). On the other hand a proper evaluation of the bosonic measure in the path integral uncovers the presence of the Wess–Zumino term. Jackiw and Rajaraman proposed the chiral Schwinger model as an example of anomalous theory (Girotti *et al.*, 1986; Jackiw and Rajaraman, 1985; Rajaraman, 1985). The model was investigated recently both from Batalin–Fradkin–Tyutin and gauge unfixing point of view (Vytheeswaran).

For these reasons the path integral quantization of the chiral Schwinger model, on the extended phase space, using Hamilton–Jacobi formulation, is interesting to investigate.

The plan of the paper is the following:

In Section 2 the multi-Hamilton–Jacobi formalism is presented. In Section 3 the chiral Schwinger model is analyzed using this formalism. The conclusions are presented in Section 4.

2. HAMILTON–JACOBI FORMALISM

This section is devoted to the presentation of the multi-Hamilton–Jacobi formulation (Güler, 1987a,b, 1989, 1992a,b). Instead of usual variational principle, a method suggested by *Carathéodory* is followed (Carathéodory, 1967). If the rank of the Hessian matrix

$$\frac{\partial^2 L}{\partial \dot{q}_i \dot{q}_j}, \quad i, j = 1, \ldots, n \tag{1}$$

is $m < n$,

$$p_a = \frac{\partial L}{\dot{q}_a}, \quad \mu = m+1, \ldots, n, \qquad p_\mu = \frac{\partial L}{\dot{q}_a}, \quad \mu = m+1, \ldots, n, \tag{2}$$

gives primary constraints

$$H'_\mu = p_\mu + H_\mu(\tau, q_i, p_a) = 0. \tag{3}$$

If the usual Hamiltonian H_0 is defined as

$$H_0 = -L + p_a \omega_a - \dot{q}_\mu H_\mu \tag{4}$$

we may solve Eqs. (2) for $\dot{q}_b$ as $\dot{q}_b = \omega_b(p_\mu, q_i, \dot{q}_\mu, p_a)$. Hence, the Hamilton–Jacobi function $S(\tau, q_i)$ should satisfy the following set of partial differential equations simultaneously for an extremum:

$$\begin{aligned} H'_0\left(\tau, q_i, p_a = \frac{\partial S}{\partial q_a}, p_0 = \frac{\partial S}{\partial \tau}\right) = 0, \\ H'_\mu\left(\tau, q_i, p_a = \frac{\partial S}{\partial q_a}, p_0 = \frac{\partial S}{\partial \tau}\right) = 0, \end{aligned} \tag{5}$$

where $H'_0 = p_0 + H_0$ and H'_μ is given by (3). So, we have multi-Hamiltonian system to start with. The canonical equations corresponding to H'_0 and H'_μ are total differential equations in t_β are as follows

$$\begin{aligned} dq_r &= \frac{\partial H'_\alpha}{\partial p_r} dt_\alpha, \\ dp_r &= -\frac{\partial H'_\alpha}{\partial q_r} dt_\alpha, \quad r = 0, 1, \ldots, n, \ \alpha = 0, 1, \ldots, m, \\ dz &= \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dt_\alpha, \quad z = S(t_\alpha, q_a), \end{aligned} \tag{6}$$

where $t_0 = \tau$, $t_\mu = q_\mu$.

Since the canonical equations are total differential equations their integrability conditions should be considered. In other words Eqs. (6) are integrable if and only if $dH'_0 = 0$, $dH'_\mu = 0$. If these variations are not zero then additional constraints may arise. Thus, we may have Hamiltonians other than initial ones. The essence of the formalism is to express all Hamiltonians as

$$H'_0 = p_0 + H_0, \qquad H'_\alpha = p_\alpha + H_\alpha. \tag{7}$$

On the other hand we know that the integrability conditions are the same as Dirac's consistency conditions (Pimentel *et al.*, 1998). Even if we recover the same results as in Dirac's formalism we cannot say that we describe the system by Hamilton–Jacobi formalism. In order to have this interpretation the Hamiltonians must be in involution and in the form given by (7).

3. THE CHIRAL SCHWINGER MODEL

The bosonized version has the following Lagrangian density (Girotti *et al.*, 1986; Jackiw and Rajaraman, 1985; Rajaraman, 1985):

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\phi)^2 + e(g^{\mu\nu} - \epsilon^{\mu\nu})(\partial_\mu\phi)A_\nu + \frac{1}{2}e^2\alpha A_\mu^2, \tag{8}$$

where $g^{\mu\nu} = \text{diag}(1, –1)$, $\epsilon^{01} = -\epsilon^{10} = 1$, and α is the regularization parameter. The lagrangian is gauge noninvariant for all values of α. We consider the case $\alpha > 1$.

The canonical Hamiltonian density is

$$\begin{aligned} H_{\mathrm{c}} = {} & \frac{1}{2}\pi_1^2 + \frac{1}{2}\pi_\phi^2 + \frac{1}{2}(\partial_1\phi)^2 + e(\partial_1\phi + \pi_\phi)A_1 + \frac{1}{2}e^2(\alpha+1)A_1^2 \\ & - A_0\left[-\partial_1\pi_1 + \frac{1}{2}e^2(\alpha-1)A_0 + e(\partial_1\phi + \pi_\phi) + e^2A_1\right], \end{aligned} \tag{9}$$

where the momenta conjugate to A_1 and ϕ are $\pi_1 = F^{01} = \partial^0A^1 - \partial^1A^0$ and $\pi_\phi = \partial_0\phi + e(A_0 - A_1)$. The canonical momentum π_0 corresponding to A_0 vanishes, and thus the Hamiltonian densities are

$$H_0' = \int (p_0 + H_{\mathrm{c}})\,dx, \tag{10}$$

$$H_1' = \int \pi_0\,dx. \tag{11}$$

Taking the variation of (10) we get

$$-\partial_1\pi_1 + e^2(\alpha-1)A_0 + e(\partial_1\phi + \pi_\phi) + e^2A_1 = 0. \tag{12}$$

Since the variation of (12) is zero then we conclude that we have three Hamiltonians

$$\begin{aligned} & H_0' = \int (p_0 + H_{\mathrm{c}})\,dx, \qquad H_1' = \int \pi_0\,dx, \\ & H_2' = \int [-\partial_1\pi_1 + e^2(\alpha-1)A_0 + e(\partial_1\phi + \pi_\phi) + e^2A_1]\,dx, \end{aligned} \tag{13}$$

which is in agreement with the corresponding Dirac's analysis. By direct calculations we can verify that the Hamiltonians given by (13) are not in involution, thus the corresponding system of total differential equations is not integrable.

Since our aim is to quantize the system presented above using the multi-Hamilton–Jacobi formalism we must transform it in such a way that the Hamiltonians (13) are in involution (Carathéodory, 1967).

For this reason we enlarge the system using Batalin–Fradkin–Tyutin formalism (Batalin and Fradkin, 1986, 1987; Batalin *et al.*, 1989a,b; Batalin and Tyutin, 1991). The essence of this formalism is to enlarge the phase space with some extra variables such that the modified canonical Hamiltonian and modified second class constraints are in involution (for more details see Batalin and Fradkin, 1986, 1987; Batalin *et al.*, 1989a,b; Batalin and Tyutin, 1991).

In our case we need only two extra fields θ, π_θ. The new Hamiltonian densities (13) are

$$\begin{aligned} H_0'' &= \int dx \left[p_0 + \frac{-\theta(e\pi_1 + e(\alpha-1)\partial_1 A_1)}{\sqrt{\alpha-1}} + \frac{e^2}{2(\alpha-1)}\theta^2 + \frac{1}{2}(\partial_1\theta)^2 \right. \\ &\quad + \frac{1}{2}\pi_\theta^2 - \frac{\pi_\theta}{e\sqrt{\alpha-1}}(-\partial_1\pi_1 + e^2(\alpha-1)A_0 + e(\partial_1\phi + \pi_\phi) + e^2 A_1) \\ &\quad \left. + \pi_\theta e\sqrt{\alpha-1} + H_c \right], \qquad (14) \\ H_1'' &= \int \left[\pi_0 + e\sqrt{\alpha-1}\theta\right] dx, \\ H_2'' &= \int \left[\pi_\theta + \frac{-\partial_1\pi_1 + e^2(\alpha-1)A_0 + e(\partial_1\phi + \pi_\phi) + e^2 A_1}{e\sqrt{\alpha-1}}\right] dx, \end{aligned}$$

where H_c is the Hamiltonian density given by (9). From (6) and (14) and taking into account that $dA_0 = \dot{A}\, d\tau$ and $d\theta_0 = \dot{\theta}\, d\tau$ we find

$$\begin{aligned} z &= \int dx\, d\tau \left[\frac{\pi_\phi^2}{2} + \frac{\pi_1^2}{2} - eA_1\partial_1\phi - \frac{1}{2}e^2(\alpha+1)A_1^2 \right. \\ &\quad + A_0\left(\frac{1}{2}e^2(\alpha-1)A_0 + e\partial_1\phi + e^2A_1\right) + e\theta\sqrt{\alpha-1}\partial_1 A_1 - \frac{e^2}{2(\alpha-1)}\theta^2 \\ &\quad - \frac{1}{2}(\partial_1\theta)^2 - \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\pi_\theta^2 + \frac{\pi_\theta}{e\sqrt{\alpha-1}}(e^2(\alpha-1)A_0 + e\partial_1\phi + e^2A_1) \\ &\quad \left. - e\theta\sqrt{\alpha-1}\dot{A}_0 - \dot{\theta}\left(e\sqrt{\alpha-1}A_0 + \frac{\partial_1\phi}{\sqrt{\alpha-1}} + \frac{eA_1}{\sqrt{\alpha-1}}\right)\right]. \qquad (15) \end{aligned}$$

Since the surface terms $\partial_1\phi$ and $\partial_1\pi_1$ from (14) give no contribution to the action z we can change the variable A_0 into A_0' as $A_0' \to A_0 + \frac{\pi_\theta}{e\sqrt{\alpha-1}} - \frac{\dot{\theta}}{e\sqrt{\alpha-1}}$. Then the action given by (15) becomes

$$\begin{aligned} z &= \int dx\, d\tau \left[\frac{\pi_\phi^2}{2} + \frac{\pi_1^2}{2} - eA_1\partial_1\phi - \frac{1}{2}e^2(\alpha+1)A_1^2 + \frac{1}{2}e^2(\alpha+1)(A_0')^2 \right. \\ &\quad + e^2A_1A_0' - \frac{e^2}{2(\alpha-1)}\theta^2 + \frac{1}{2}\partial_\mu\partial^\mu\theta + e\partial_1\phi(A_0' - A_1) + e\theta\sqrt{\alpha-1}\partial_1A_1 \\ &\quad \left. + \dot{\theta}e\sqrt{\alpha-1}A_0' - \frac{1}{2}(\partial_1\phi)^2 \right]. \qquad (16) \end{aligned}$$

Making the transformation $\pi_1 \mapsto \pi_1' = \pi_1 - \frac{e\theta}{\sqrt{(\alpha-1)}}$, $\theta \mapsto \theta\sqrt{\alpha-1}$, and $\tau \mapsto t$, the action becomes

$$z = \int dx\, d\tau \left[\frac{-1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{e^2\alpha}{2} A_\mu A^\mu + e(g^{\mu\nu} - \epsilon^{\mu\nu})(\partial_\mu\phi)A_\nu + \frac{1}{2}(\partial_\mu\phi)^2 \right.$$
$$\left. + \frac{\alpha-1}{2}(\partial_\mu\theta)^2 - e\theta((\alpha-1)g^{\mu\nu} + \epsilon^{\mu\nu})(\partial_\mu A_\nu) \right]. \tag{17}$$

The action (17) is the same as obtained by Wess and Zumino (1971).

4. CONCLUDING REMARKS

In this paper we quantized the bosonized version of the chiral Schwinger model using Hamilton–Jacobi formalism.

We find the same constraints as in Dirac's formalism and we conclude that the corresponding system of total differential equations is not integrable since the Hamiltonian densities are not in involution.

By enlarging the phase space the system of three Hamiltonian densities becomes in involution and in the form given by (7).

Surface terms are essential to recover the same result as obtained by adding the Wess–Zumino terms to the original bosonized action (8).

For a given nonintegrable system from multi-Hamilton–Jacobi point of view we can associate, depending on the method used for making Hamiltonians in involution, an integrable system.

Using Batalin–Fradkin–Tyutin formalism for converting the second-class constraints into first-class constraints it is not the unique procedure to get a valid multi-Hamilton–Jacobi formulation. In other words we can associate different integrable multi-Hamilton–Jacobi systems if we use different approaches of abelianization of second-class constraints.

An interesting question is to use the gauge unfixing method (Annishetty and Vytheeswaran, 1993; Vytheeswaran, 1994) for making the constraints in involution and then to analyze the characteristics of the corresponding multi-Hamilton–Jacobi system. This problem is under investigation (Baleanu and Güler, manuscript in preparation).

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REFERENCES

Annishetty, R. and Vytheeswaran, A. S. (1993). *Journal of Physics A: Mathematical and General* **26**, 5613.
Baleanu, D. and Güler, Y. (2000). *Nuovo Cimento B* **115**, 25.
Baleanu, D. and Güler, Y. (2001). *Journal of Physics A: Mathematical and General* **34**, 73.
Baleanu, D. and Güler, Y. (2001). *International Journal of Modern Physics A* **16**, 2391.
Batalin, I. A. and Fradkin, E. S. (1986). *Physics Letters B* **180**, 156.
Batalin, I. A. and Fradkin, E. S. (1987). *Nuclear Physics B* **279**, 514.
Batalin, I. A., Fradkin, E. S., and Fradkina, E. S. (1989a). *Nuclear Physics B* **314**, 158.
Batalin, I. A., Fradkin, E. S., and Fradkina, E. S. (1989b). *Nuclear Physics B* **323**, 734.
Batalin, I. A. and Tyutin, I. V. (1991). *International Journal of Modern Physics A* **6**, 3255.
Bergmann, P. G. (1966). *Physical Review D* **144**, 1078.
Carathéodory, C. (1922). *Mathematical Annals* **85**, 78.
Carathéodory, C. (1929). *Acta Sci. Math.* (Szeged) **4**, 193. (Reprinted in Carathéodory, C. (1954). Gesamm. Math. Schriften, Bd 1, C. H. Beck, München, p. 401.)
Carathéodory, C. (1967). *Calculus of Variations and Partial Differential Equations of First Order*, Part II (Holden-Day).
Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, New York.
Dominici, D., Gomis, J., Longhi, G., and Pons, J. M. (1984). *Journal of Mathematical Physics* **25**, 2439.
Faddeev, L. D. and Shatashvili, S. I. (1986). *Physics Letters B* **167**, 225.
Girotti, H., Rothe, K., and Rothe, H. (1986). *Physical Review D: Particles and Fields* **33**, 574.
Goldberg, J., Newman, E., and Rovelli, C. (1991). *Journal of Mathematical Physics* **32**, 2739.
Güler, Y. (1987a). *Nuovo Cimento B* **100**, 251.
Güler, Y. (1987b). *Nuovo Cimento B* **100**, 267.
Güler, Y. (1989). *Journal of Mathematical Physics* **30**, 785.
Güler, Y. (1992a). *Nuovo Cimento B* **107**, 1139.
Güler, Y. (1992b). *Nuovo Cimento B* **107**, 1143.
Güler, Y. (1998). *Nuovo Cimento B* **113**, 893.
Hanson, A., Regge, T., and Teitelboim, C. (1976). *Constrained Hamiltonian Systems*, Academia Nationale dei Lincei, Rome.
Henneaux, M. (1985). *Physics Reports* **126**, 1.
Henneaux, M. and Teiteilbom, C. (1992). *Quantization of Gauge Systems*, Princeton University Press, Princeton, NJ.
Henneaux, M., Teilteilboim, C., and Vergara, J. D. (1992). *Nuclear Physics B* **387**, 391.
Hosotani, Y. and Nakayama, R. (1999). *Modern Physics Letters A* **14**, 1983.
Jackiw, R. and Rajaraman, R. (1985). *Physical Review Letters* **54**, 1219.
Kastrup, A. H. (1977). *Physics Letters B* **70**, 195.
Kastrup, H. A. (1979). *Physics Letters B* **82**, 237.
Kastrup, H. A. (1983). *Physics Reports* **101** (1/2), 1.
Kastrup, H. A. and Rinke, M. (1981). *Physics Letters B* **105**, 101.
Komar, A. (1978). *Physical Review D* **18**, 1881.
Kuchar, K. (1982). In *Quantum Gravity, Vol. 2*, C. J. Isham, R. Penrose, and D. V. Sciama, eds., Oxford University Press, Oxford.
Nambu, Y. (1980). *Physics Letters B* **92**, 327.
Nambu, Y. (1981). *Physics Letters B* **102**, 149.
Pimentel, B. M., Teixeira, R. G., and Tomazelli, J. L. (1996). *Nuovo Cimento B* **111**, 841.
Pimentel, B. M, Teixeira, R. G., and Tomazelli, J. L. (1998). *Annals of Physics (New York)* **267**, 75.

Pimentel, B. M. and Teixeira, R. G. (1998). *Nuovo Cimento B* **113**, 805.
Rabei, E. and Güler, Y. (1992). *Physical Review A* **46**, 3513.
Rajaraman, R. (1985). *Physics Letters B* **54**, 305.
Rajaraman, R. and Mitra, P. (1990a). *Annals of Physics* (*New York*) **203**, 137.
Rajaraman, R. and Mitra, P. (1990b). *Annals of Physics* (*New York*) **203**, 157.
Shabanov, S. (2000). *Physics Reports* **326**, 1.
Sundermeyer, K. (1982). *Constrained Dynamics, Lecture Notes in Physics*, Springer, New York, Vol. 169.
Vytheeswaran, A. S. (1994). *Annals of Physics* (*New York*) **236**, 297.
Vytheeswaran, A. S. hep-th/9904084.
Wess, J. and Zumino, B. (1971). *Physics Letters B* **37**, 95.
Witten, E. (1981). *Nuclear Physics B* **188**, 513.